IMPACT OF A CONE ON A THIN ELASTIC MEMBRANE

(OB UDARE KONUSOM PO TONKOI UPRUGOI MEMBRANE)

PMM Vol.30, № 6, 1966, pp.1035-1041

S.S.GRIGORIAN and D.M.GRIGORIAN
(Moscow)

(Received August 12, 1966)

The problem of computation of parameters of motion of a thin elastic membrane under the impact of a rigid body, was considered by various authors (see [1] together with bibliography) without, however, yielding a rational solution. This paper presents a full qualitative analysis of solution of this problem for the case of normal impact of a circular cone moving with constant velocity on an infinite elastic membrane of constant thickness. Although this is the simplest case, it is important, insofar as it brings to light the characteristic features of the problem. In the "membrane" approximation the thickness of the layer is found to be an unessential parameter, therefore the problem, as postulated by us, is self-similar and its solution is reducible to the problem for ordinary differential equations.

In the formulation of our problem we shall use the basic relations given in [1]. We shall take the plane $y = 0$ in the cylindrical coordinate system r, φ, y to represent the middle surface of the membrane and we shall assume the cone to be moving along the oy-axis with constant velocity v_0. Then, the equations of motion will be

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(\sigma_t r \cos\gamma) - \frac{\sigma_\varphi}{r}, \quad \rho\frac{\partial^2 u}{\partial t^2} = -\frac{1}{r}\frac{\partial}{\partial r}(\sigma_t r \sin\gamma) \qquad (1)$$

$$\sin\gamma = -\frac{1}{1+e_t}\frac{\partial u}{\partial r}, \qquad \cos\gamma = \frac{1}{1+e_t}\left(1+\frac{\partial w}{\partial r}\right) \qquad (2)$$

where u and w are the displacements of the points of the middle surface in the y- and r-directions, respectively; r is the Lagrangian coordinate of a point on the middle surface of the membrane, t is the time, ρ is the density of the membrane material, σ_t and σ_φ are the stress components referred to the initial areas of boundaries at which they are applied and strain components are given by

$$e_t = \left[\left(1+\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r}\right)^2\right]^{1/2} - 1, \qquad e_\varphi = \frac{w}{r} \qquad (3)$$

For the elastic membrane we have

$$\sigma_t = \frac{E}{1-\nu^2}(e_t + \nu e_\varphi), \qquad \sigma_\varphi = \frac{E}{1-\nu^2}(e_\varphi + \nu e_t) \qquad (4)$$

where E and ν are the Young's modulus and Poisson's coefficient, respectively. In fact, the relations (4) define, in the present case, the elastic law.

Elimination of stress and strain from (1) to (4) yields two equations for u and w. Supplementary initial conditions are

$$u(r, 0) = w(r, 0) = 0 \tag{5}$$

and the boundary condition at the point of impact

$$w(0, t) = 0, \qquad u(0, t) = v_0 t \tag{6}$$

In addition, functions u and w should obviously satisfy the kinematic conditions resulting from the possible relative positioning of the membrane and the cone (the membrane may partially envelope the cone, with the rest remaining outside the cone). Unknown functions u and w depend therefore on the independent variables r and t and on the parameters θ, v_0, E, ν and ρ (2θ is the cone angle) of the problem, which is, consequently, self-similar and has a solution of the form

$$u = a_0 t Y(z; \nu, m, \theta_0), \qquad w + r = a_0 t X(z; \nu, m, \theta_0)$$

$$z = \frac{r}{a_0 t}, \quad a_0 = \left(\frac{E}{\rho(1-\nu^2)}\right)^{1/2}, \quad m = \frac{v_0}{a_0} \tag{7}$$

In this manner our problem is reduced to the determination of functions X and Y satisfying the corresponding ordinary differential equations and boundary conditions. It can be shown that, in general, three distinct regions of variation of z exist, and the above functions are defined in each of these regions in a different way. We shall not be concerned with the outermost region $r > a_0 t$, i.e. $z > 1$ where $w \equiv u \equiv 0$. In the region $z_* \leq z \leq 1$ we have [1]

$$Y \equiv 0, \qquad X = z - c\omega_1(z), \qquad c = \text{const}$$

$$\omega_1(z) \equiv \ln\frac{1+\sqrt{1-z^2}}{z} - \frac{\sqrt{1-z^2}}{z^2} \qquad (z_* \leqslant z \leqslant 1) \tag{8}$$

This is the region of purely radial motion of the elements of the membrane, and its outer boundary is formed by the elastic wavefront $z = 1$. Its inner boundary $z = z_*$ and the constant c are not given in advance and should be determined in the course of solution of the problem. In the region $z_{**} \leq z \leq z_*$ where the membrane is deflected, without however coming into contact with the surface of the cone, functions X and Y are the solution of

$$X'' = \frac{z\,[(1+\nu)z - \nu X]\,X'R' - z(1+\nu-R)\,RX' + \nu zR(X')^2 + R^2[(1+\nu)z - \nu zR - X]}{zR\,\{[1+\nu-R(1-z^2)]\,z - \nu X\}}$$

$$R' = \frac{[X-(1+\nu)z]\,X' + zR(1+\nu-R)}{z^2(1-z^2)\,R}, \qquad Y' = \sqrt{R^2-(X')^2} \tag{9}$$

satisfying some boundary conditions which shall be specified later, at the points $z = z_*$ and $z = z_{**}$. Equations (9) ar obtained from (1) to (4) and (7). The magnitude $z = z_{**}$ defining the point of contact between the membrane and the cone, is also unknown.

Finally, in the region $0 \leq z \leq z_{**}$ the membrane envelopes the surface of the cone and functions X and Y are given by

$$X = U(z) \sin\theta, \qquad Y = m - U(z)\cos\theta, \qquad R = U'(z)$$

$$U(z) = \frac{1+\nu}{1+\lambda} z + c_1 z^\lambda F\left(\frac{\lambda}{2},\ \frac{\lambda-1}{2},\ \lambda+1;\ z^2\right) \tag{10}$$

where $\lambda = \sin\theta$, $c_1 = \text{const}$, and F is a hypergeometric function [1]

$$F(\alpha, \beta, \gamma; x) = 1 + \frac{\alpha\beta}{1!\gamma} x + \ldots \tag{11}$$

Here it is the constant c_1 which is the unknown parameter to be determined. In this manner, the sought functions X and Y are defined in the regions $z_* \leq z \leq 1$ and $0 \leq z \leq z_{**}$ by the finite formulas (8),(10) and (11) respectively, and by ordinary differential equations (9) in the intermediate region $z_{**} \leq z \leq z_*$. The choice of the solution of (9) and the determination of parameters z_{**}, z_*, c and c_1 should be performed with the aid of conditions of compatibility at the points $z = z_*$ and $z = z_{**}$. Obviously, the most suitable conditions will be the conditions of continuity of X and Y . Further, the usual conservation laws (of mass and momentum) should be preserved at these points, which correspond, physically, to the expanding surfaces of discontinuity.

Let us first consider the conditions at the point $z = z_*$. Using the subscripts 1 and 2 to denote the values at $z < z_*$ and $z > z_*$, respectively, we obtain the condition of conservation of mass

$$\frac{\beta - X_2 + z_* X_2'}{R_2} = \frac{(\beta - X_1 + z_* X_1')\cos\gamma + (Y_1 - z_* Y_1')\sin\gamma}{R_1} \tag{12}$$

and the momentum theorem

$$(\beta - X_2 + z_* X_2')(X_2' - X_1')\frac{z_*}{R_2} = R_2 + \frac{\nu}{z_*} X_2 - \left[R_1 + \frac{\nu}{z_*} X_1 - (1+\nu)\right]\cos\gamma \qquad \left(\beta = \frac{b}{a_0}\right) \tag{13}$$

$$-(\beta - X_2 + z_* X_2') Y_1' \frac{z_*}{R_2} = \left[R_1 + \frac{\nu}{z_*} X_1 - (1+\nu)\right]\sin\gamma \tag{14}$$

where b is the velocity of propagation of the discontinuity along the or-axis and γ is the angle between the meridional elements of the membrane on both sides of the discontinuity. Supplementing (12) to (14) we have the obvious equalities

$$X_1 = X_2, \quad Y_1 = 0, \quad \beta = X_2, \quad R_2 = X_2' \tag{15}$$

together with the kinematic relationships

$$\sin\gamma = -\frac{Y_1'}{R_1}, \qquad \cos\gamma = \frac{X_1'}{R_1} \tag{16}$$

Condition (12) becomes, by virtue of (15) and (16), an identity, while (13) and (14), together with (8) and the condition $Y' \neq 0$, yield

$$R_1(z_*) = \frac{\nu X_2(z_*) - (1+\nu) z_*}{z_*(z_*^2 - 1)} \tag{17}$$

$$X_1(z_*) = X_2(z_*) = z_* - c(z_*)\,\omega_1(z_*) \tag{18}$$

$$c(z_*) = \frac{z_*^2}{\omega_1'(z_*)(z_*^2 - 1) - \nu z_*^{-1}\omega_1(z_*)} \tag{19}$$

In this manner, we have expressed all the unknowns at the point $z = z_*$, in terms of z_*. Formula (19) gives the parameter c from (8) in terms of z_*, while the relations (17) and (18), together with the condition $Y_1(z_*)=0$, fully define at $z = z_*$, the conditions of transition from (8) to the solution of (9), which in turn defines the solution of our problem in the region $z_{**} \leq z \leq z_*$. Next we shall consider the conditions at the point $z = z_{**}$.

Equation of the conservation of mass has, in dimensionless variables, the form

$$\frac{(\beta_r - X_1 + z_{**}X_1')\sin\theta_1 - (\beta_y - Y_1 + z_{**}Y_1')\cos\theta_1}{R_1} = \frac{(\beta_r - X_2 + z_{**}X_2')\sin\theta_2 - (\beta_y - Y_2 + z_{**}Y_2')\cos\theta_2}{R_2} \tag{20}$$

Here subscripts 1 and 2 denote the limiting values of the corresponding magnitudes on both sides of the surface of discontinuity and θ_1 and θ_2 are the angles of inclination of the elements of the membrane on both sides of the discontinuity, towards the or-axis. In addition, we have the conditions of continuity of displacements

$$X_1 = X_2, \qquad Y_1 = Y_2 \tag{21}$$

the relations for the components β_r and β_y of the velocity of the surface of discontinuity

$$\beta_r = X_1(z_{**}), \qquad \beta_y = Y(z_{**}) \tag{22}$$

and the kinematic relationships

$$\frac{X_1'}{R_1} = \sin\theta_1, \quad \frac{Y_1'}{R_1} = -\cos\theta_1, \quad \frac{X_2'}{R_2} = \sin\theta_2, \quad \frac{Y_2'}{R_2} = -\cos\theta_2 \tag{23}$$

Obviously, in this case the relations (21) to (23) make (20) an identity, and the actual conditions at the point $z = z_{**}$ are given only by the momentum theorem (with (21) to (23) taken, naturally, into account)

$$z_{**}^2(X_1' - X_2') = \frac{X_1'}{R_1}\left(R_1 + \nu\frac{X_1}{z_{**}} - 1 - \nu\right) - \frac{X_2'}{R_2}\left(R_2 + \nu\frac{X_2}{z_{**}} - 1 - \nu\right) + \frac{X_1}{z_{**}}q_r \tag{24}$$

$$z_{**}^2(Y_1' - Y_2') = \frac{Y_1'}{R_1}\left(R_1 + \nu\frac{X_1}{z_{**}} - 1 - \nu\right) - \frac{Y_2'}{R_2}\left(R_2 + \nu\frac{X_2}{z_{**}} - 1 - \nu\right) + \frac{X_1}{z_{**}}q_y \tag{25}$$

$$q_r = \frac{Q_r}{\rho a_0^2\delta}, \qquad q_y = \frac{Q_y}{\rho a_0^2\delta} \tag{26}$$

Here Q_r and Q_y are the components of a concentrated force, which may appear at the point where the membrane separates the surface of the cone and δ is the initial thickness of the membrane. In the present case, we must put $Q_r = Q_y = 0$ when $z_{**} < z_*$. Taking that, together with

$$Y' = -\sqrt{R^2 - (X')^2},$$

into account, we can reduce (24) and (25) to

$$X_1'\left[z_{**}^2 - 1 - \nu\frac{X_1}{R_1 z_{**}} + \frac{1+\nu}{R_1}\right] = X_2'\left[z_{**}^2 - 1 - \nu\frac{X_2}{R_2 z_{**}} + \frac{1+\nu}{R_2}\right] \quad (27)$$

$$R_1\left[z_{**}^2 - 1 - \nu\frac{X_1}{R_1 z_{**}} + \frac{1+\nu}{R_1}\right] = R_2\left[z_{**}^2 - 1 - \nu\frac{X_2}{R_2 z_{**}} + \frac{1+\nu}{R_2}\right] \quad (28)$$

From (28) it follows that $R_1 = R_2$, and (27) implies the continuity of X' and hence of Y', i.e. $X_1' = X_2'$ and $Y_1' = Y_2'$, provided that the expressions within the square brackets are different from zero. Since they can become zero only fortuitously, we can draw from the conditions at the point $z = z_{**}$, a general conclusion, that at this point the functions X, X', Y, Y' and R are continuous, i.e. only a weak discontinuity occurs at the point where the membrane separates from the surface of the cone.

Hence, the required solution of (9) which, together with (8) and (10) will fully define the solution of our problem, should be constructed according to the conditions (17) and (18) at the point $z = z_*$ and the condition of continuous compatibility with Formulas (10) at the point $z = z_{**}$.

Let us fix the values of θ and ν. Then, c_1 will be the only undetermined element in (10).

Assigning to it some value and assuming that $z = z_{**}$, we find, that the values of X, X' and R at the point $z = z_{**}$ which are given uniquely by (10), supply a full compliment of initial values for first two equations of (9). Numerical solution of the resulting Cauchy problem leads to the determination of the points $z = z_{*X}$ and $z = z_{*R}$, at which the curves of constructed functions $X = X(z)$ and $R = R(z)$ intersect the lines given by (17) and (18). In general, we find that $z_{*X} \neq z_{*R}$. In this case we alter z_{**} and repeat the procedure until the condition $z_{*X} = z_{*R}$. is satisfied. From this we can see, that our numerical solution is based on determination of the root z_{**} of Equation

$$f(z_{**}) \equiv z_{*X}(z_{**};\ \theta, c_1, \nu) - z_{*R}(z_{**};\ \theta, c_1, \nu) = 0 \quad (29)$$

which defines the values z_{**} and z_* together with the functions X and R everywhere within the interval $0 \le z \le 1$. To complete the solution, we must compute the quadrature

$$Y(z) = -\int_{z_*}^{z} \sqrt{R^2 - (X')^2}\, dz \quad (30)$$

Conditions of continuity of Y at the point $z = z_{**}$ leads, together with (30) and (10), to

$$Y(z_{**}) = -\int_{z_*}^{z_{**}} \sqrt{R^2 - (X')^2}\, dz = m - U(z_{**})\cos\theta \quad (31)$$

which yields the value of the dimensionless velocity m of the cone, corresponding to the assumed values of θ, ν and c_1.

$$m = U(z_{**})\cos\theta + \int_{z_{**}}^{z_*} \sqrt{R^2 - (X')^2}\, dz \tag{32}$$

These computations can be performed for various values of the parameter c_1 with resulting solutions corresponding to various impact velocities. However, the range of possible variation of c_1 over which the above scheme of motion is realized, must be found.

The upper boundary of this range is obviously given by the condition $z_{**} = z_*$, while the lower boundary, by $z_{**} = 0$.

It can be shown that when $\theta \neq 0$, then the condition $z_{**} = 0$ cannot be achieved irrespective of how small the impact velocity is, i.e. at any impact velocity, part of the membrane will envelope the cone, no matter how thin the latter is. This condition emerges from the consideration of the asymptotic behavior of solutions of (9) when $z \to 0$. If, at some impact velocity and some value of θ, $z_{**} = 0$ and the membrane was in contact with the cone at one point only, namely at the vertex of the cone, then the solution of (9) would exist which would, in the vicinity of the point $z = 0$, exibit the asymptotic property $Y = aX'$, $a = \text{const} < 0$. Analysis of Equations (9) shows that such an asymptotic behavior is possible only when $a = -\infty$. This proves the above assertion and at the same time implies, that in the case of the point impact ($\theta = 0$) at the membrane with constant velocity, meridional cross section of the deformed membrane exhibits a cusp at the point of impact.

Consequently, the lower boundary of the range of variation of c_1 is equal to zero, since the solution of (10) tends to zero as $c_1 \to 0$ although, when $\theta \neq 0$, it is contained in the complete solution of the problem for arbitrary impact velocity and tends to zero together with this velocity, i.e. when $m \to 0$, $c_1 \to 0$.

Let us now determine the upper boundary of the range of variation of c_1 . This is easily done by inserting the values of $X_1(z_*)$ and $R_1(z_*)$ given by (10), into (17) and (18). This is equivalent to the condition $z_{**} = z_*$ defining $\max c_1$. As a result, we obtain two equations defining $\max c_1$ and the corresponding value of $z_* = z_{**} = z_{*0}$. The respective formulas are

$$\frac{X_2(z_{*0}) - (a\sin\theta)\, z_{*0}}{Q(z_{*0})\sin\theta} = \frac{R_2(z_{*0}) - a}{Q'(z_{*0})}$$

$$Q(z) = z^\lambda F\left(\frac{\lambda}{2}, \frac{\lambda - 1}{2}, \lambda + 1; z^2\right), \quad a = \frac{1+\nu}{1+\lambda}, \quad \lambda = \sin\theta \tag{33}$$

$$X_2(z) = z\left[1 - \frac{z^2}{z(1-z^2)(\omega_2(z)/(\omega_1(z)) - \nu}\right], \quad R_2(z) = \left[1 + \frac{z^3}{\nu\,\omega_2(z)/\omega_1(z) - z}\right]^{-1}$$

$$\omega_2(z) = -\omega_1'(z) = \frac{\sqrt{1-z^2}}{z} + \ln\frac{1+\sqrt{1-z^2}}{z} \tag{34}$$

$$\max c_1 = \frac{R_2(z_{*0}) - a}{Q'(z_{*0})} = \frac{X_2(z_{*0}) - (a\sin\theta)\, z_{*0}}{Q(z_{*0})\sin\theta}$$

Asymptotic formulas may be found useful in constructing the solution for small θ. Letting θ in (33) and (34) tend to zero, we obtain the asymptotic equation for z_{*0}

$$\psi(z_{*00}) X_2(z_{*00}) = z_{*00}[R_2(z_{*00}) - (1+\nu)]$$

$$\psi(z) \equiv \lim_{\theta \to 0} \frac{Q'(z;\theta)}{\theta Q(z;\theta)}, \qquad c_1|_{\theta\to 0} \to \frac{X_2(z_{*00})}{\theta} \tag{35}$$

The obtained result is interesting: as $\theta \to 0$, z_{*0} tends to the limiting value z_{*00}, i.e. the coordinate of the point at which the membrane separates from the surface of the cone is, in the absence of the intermediate region $z_{**} = z_*$, independent of θ for small values of θ. Naturally, the impact velocity increases with decreasing θ, because $c_1 \to \infty$ and m behaves similarly to c_1 (see (22)).

The procedure described above allows us to solve the problem for $c_1 < \max c_1$ and for impact velocities m not exceeding $m = m(\max c_1)$ At high impact velocities, the mode of motion corresponds to the complete contact with the cone over the interval $0 \le z \le z_*$ and over the region of radial motion $z_* \le z \le 1$, but here the components of the localized force q_r and q_y will no longer be equal to zero. Solution in the region $0 \le z \le z_*$ will, as before, be defined by Formulas (10) and in the region $z_* \le z \le 1$, by Formulas (8); constants c and c_1 and the value of z_* should, however, be found by another method. Condition $Y_1(z_*) = Y_2(z_*) = 0$ yields $U(z_*)\cos\theta = m$ and this makes it possible to express the parameter c_1 in terms of m and z_* in the form

$$c_1 = \bar{c}_1(z_*) = \left(\frac{m}{\sqrt{1-\lambda^2}} - \frac{1+\nu}{1+\lambda} z_*\right) \Big/ z_*^{\lambda} F\left(\frac{\lambda}{2}, \frac{\lambda-1}{2}, \lambda+1; z_*^2\right) \tag{36}$$

Condition $X_1(z_*) = X_2(z_*)$ together with (8) and (10), yields

$$c = \bar{c}(z_*) = \frac{z_* - m\tan\theta}{\omega_1(z_*)} \tag{37}$$

Equation of conservation of mass at $z = z_*$ is, as we noted before, fulfilled automatically and it only remains to satisfy the momentum theorem, i.e. the relations (24) and (25) in which z_{**} is replaced with z_*. Then, these relations together with (8),(10),(36) and (37) allow us to express q_r and q_y in terms of z_*. If we denote by φ the angle of friction between the materials of the membrane and the cone, i.e. $\varphi \equiv \tan^{-1} f$ where f is the coefficient of friction between the membrane and the cone, then, assuming that the localized force (q_r, q_y) makes the angle φ with the normal, we obtain the relation

$$q_y = q_r \tan(\theta - \varphi) \tag{38}$$

Substituting into it the above-mentioned expressions for q_r and q_y in terms of z_*, we obtain the equation defining z_* and upon solving it, we arrive at the complete solution of the problem for the corresponding values of velocities within the considered interval. At the same time it should be checked whether slipping of the membrane on the cone at the point $z = z_*$

occurs, i.e. whether the condition $U(z_*) - z_* U'(z_*) \neq 0$ holds. If, at some impact velocity the conditions (38) and

$$U(z_*) - z_* U'(z_*) = 0 \tag{39}$$

are fulfilled simultaneously, then this velocity divides the velocity range under consideration into two parts, in one of which slipping takes place and (38) holds, while in the other slipping is absent and (39) holds. Critical value of z_* is easily found from the available formulas. It seems, that the motion without slipping occurs first and takes place up to the critical velocity, after which we have the motion with slipping.

The above scheme defines the solution for the range of velocities over which $z_* < 1$. When $m = \cot\theta$, we have $z_* = 1$, and the solution should be constructed in a different way when $m > \cot\theta$. In this case the region of radial motion cannot exist and we have $z_* > 1$, the value of which is found simply from $z_* = m\tan\theta$. This yields a single discontinuity $z = z_*$ and the problem cannot be solved, unless another discontinuity $z = z_{**} < z_*$ is introduced in the region where the solution is given by Formulas of the type (10). This is completely analogous to the development occurring in the problem on the impact of a wedge on a thread [1].

BIBLIOGRAPHY

1. Dem'ianov, Iu.A. and Rakhmatulin, Kh.A., Prochnost' pri intensivnykh kratkovremennykh nagruzkakh (Strength of Materials under Strong Impulsive Loads). M., Fizmatgiz, 1961.

Translated by L.K.